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# Exercises for 'Topics in complex analysis'

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**H 12.1 (The Riemann sphere $\widehat{\mathbb{C}}$)**
The goal of this exercise is to introduce calculus at $\infty$. We set $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where for the moment $\infty$ is an abstract element. We say that a sequence $\{z_n\} \subset \widehat{\mathbb{C}}$ converges to $z \in \widehat{\mathbb{C}}$ if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\begin{cases} z_n = \infty \quad \text{or} \quad |z_n| \geq \varepsilon^{-1} & \text{if } z = \infty, \\ z_n \in \mathbb{C} \quad \text{and} \quad |z_n - z| \leq \varepsilon & \text{if } z \in \mathbb{C}. \end{cases}$$

Set $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$. Show that the stereographic projection $P : \mathbb{S}^2 \to \widehat{\mathbb{C}}$ defined by

$$P(x_1, x_2, x_3) = \begin{cases} \frac{x_1}{1-x_3} + i\frac{x_2}{1-x_3} & \text{if } x_3 \neq 1, \\ \infty & \text{otherwise} \end{cases}$$

is bijective, and that both $P$ and its inverse $P^{-1}$ are (sequentially) continuous. Conclude that $\widehat{\mathbb{C}}$ is (sequentially) compact.

**Remark**: Observe that the definition of convergence given above coincides with the one induced by the topology of $\widehat{\mathbb{C}}$ described in the lecture notes.

**H 12.2 (The open mapping theorem for $\widehat{\mathbb{C}}$)**
Let $\widehat{D} \subset \widehat{\mathbb{C}}$ be a domain and let $f : \widehat{D} \to \widehat{\mathbb{C}}$ be holomorphic and non-constant. Show that $f(\widehat{D})$ is again a domain.

**H 12.3 (On the extension of entire functions)**
In this exercise we prove that polynomials are the only entire functions that have a holomorphic extension to the Riemann sphere.

a) Let $P : \mathbb{C} \to \mathbb{C}$ be a non-constant polynomial. Show that setting $P(\infty) := \infty$ defines a holomorphic extension $P : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

b) Show that if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic and satisfies $f(\mathbb{C}) \subset \mathbb{C}$ then $f$ is a polynomial.
**Hint**: Consider $z \mapsto f(1/z)$ and its singularity at 0.

**H 12.4 (Holomorphic functions on $\widehat{\mathbb{C}}$ are rational)**
Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be holomorphic. Show that $f$ is a rational function, i.e. that there exist polynomials $P, Q : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) = \frac{P(z)}{Q(z)} \qquad \forall z \in \mathbb{C} \setminus \{f = \infty\}.$$

**Hint**: You may need the following generalization of Liouville's theorem: if $g : \mathbb{C} \to \mathbb{C}$ is holomorphic and there exist $R > 0$ and $n \in \mathbb{N}$ such that $|g(z)| \leq R|z|^n$ for all $z \in \mathbb{C}$ with $|z| \geq R$, then $g$ is a polynomial of degree at most $n$. We showed an even stronger version of this result in the proof of Hadamard's theorem.